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SYMMETRIES OF CYLINDRICAL EQUATIONS OF HYDRODYNAMICS AND CONVECTIVE HEAT EXCHANGE

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Consideration has been given to the interrelation of the symmetries (in terms of the Lie groups) of a system of Navier–Stokes and Fourier–Kirchhoff equations and self-similar forms of these equations. It has been shown how various self-similar forms can be determined on the basis of the symmetries. Ordinary self-similar differential equations have been obtained.

The symmetries of differential equations (Lie groups) enable one to make an analytical study or a numerical analysis of physical processes that obey these differential equations much easier. The symmetry properties are closely related to the self-similarity property [1]. By employing the symmetries one can reduce a system of partial differential equations to an ordinary differential equation or decrease the number of independent variables. The self-similar forms of variables are also useful for experimental investigations since they point to a means of generalizing experimental data.

Investigations of the laminar and turbulent boundary layers in which the self-similar forms of Prandtl equations were obtained on the basis of the theory of Lie groups have been carried out in [2–4]. These investigations illustrate the application of the method of symmetries to the problems of hydromechanics and heat exchange.

In a number of works devoted to investigation of the symmetries of Navier–Stokes equations [5–9] or a system of Navier–Stokes equations and equations of convective heat exchange [10] with account for the Archimedes force in a Cartesian coordinate system, the symmetry groups of these equations have been found. However, there are virtually no works devoted to investigation of the symmetries of the equations of hydrodynamics and convective heat exchange in cylindrical coordinates.

The system of Navier–Stokes equations and equations of convective heat exchange in cylindrical coordinates for an incompressible fluid is as follows:

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+\frac{v}{r}\frac{\partial w}{\partial \varphi}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2}+\frac{\partial^2 w}{\partial z^2}\right)+g\beta T\,,$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial \varphi}-\frac{v^2}{r}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}-\frac{2}{r^2}\frac{\partial v}{\partial \varphi}+\frac{\partial^2 u}{\partial z^2}\right),$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial \varphi}+\frac{uv}{r}+w\frac{\partial v}{\partial z}=-\frac{1}{r\rho}\frac{\partial p}{\partial \varphi}+\nu\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \varphi^2}+\frac{2}{r^2}\frac{\partial u}{\partial \varphi}+\frac{\partial^2 v}{\partial z^2}\right), \tag{1}$$

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial \varphi}+\frac{\partial w}{\partial z}=0\,,$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial r}+\frac{v}{r}\frac{\partial T}{\partial \varphi}+w\frac{\partial T}{\partial z}=a\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \varphi^2}+\frac{\partial^2 T}{\partial z^2}\right).$$

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System (1) possesses symmetries which can be characterized by the infinitesimal generator

$$q = [C_1 2t + C_6]\,\partial_t + C_2\partial_\varphi + C_1 r\partial_r + [C_1 z + C_3 + C_7 z]\,\partial_z - C_1 u\partial_u + [C_7 w - C_1 w]\,\partial_w - C_1 v\partial_v +$$

$$+ [C_5/(g\beta\rho) - C_1 NT + C_7 T]\,\partial_T + [C_4\tau(t) - C_1 2p + C_5 z]\,\partial_p\,. \tag{2}$$

The general methods of construction of the infinitesimal generator have been described in [1, 11] in detail. In formula (2), a subscript on the symbol ∂ means a derivative with respect to the quantity presented in this subscript. The constant N is equal to 3 in the case where natural convection is substantial, i.e., where it is necessary to take into account the last term in the first equation of system (1). If this term can be disregarded, N can take on any values.

The infinitesimal generator (2) possesses the property that, based on it, one can find such transformations which do not change the form of the differential equations (1). These transformations enable one to construct new solutions of the system of equations from its known solutions [1]. Another remarkable property of the infinitesimal generator is that by employing it one can find different self-similar forms of the system of differential equations.

The constants with a numerical subscript in expression (2) characterize different types of symmetries. Thus, system (1) possesses seven symmetries. Let us briefly characterize them. The symmetry C_1 describes extension, i.e., shows how the dependent quantities (velocity, pressure, temperature) are transformed with variation in the independent (time, coordinates) arguments. The symmetries C_2 and C_3 represent shifts along the coordinate axes. The transformations of solutions on their basis describe passage to an arbitrary moving coordinate system. The symmetry C_4 reflects the fact that the pressure is determined accurate to only an arbitrary time function. The symmetry C_5 is due to the Archimedes force. It relates a constant "shift" of the temperature field to a linearly changing "shift" of the pressure field in the direction of the axis of action of the Archimedes force. The symmetry C_6 is the simplest symmetry of a time shift. The symmetry C_7 is the symmetry of reflection when the coordinate and the corresponding velocity component simultaneously change their sign. This is a discrete symmetry, in essence. It differs from the remaining symmetries of system (1), which are continuous. However, formally, discrete symmetries can also be characterized by the infinitesimal generator as extension groups but with discrete values of a multiplicative transformation parameter. Consequently, these symmetries can be employed to construct self-similar forms.

Let us consider the symmetries generated by the infinitesimal generator q_{1-6}. To construct self-similar coordinates we must write, on the basis of q_{1-6}, a partial equation containing only independent variables [1], i.e.,

$$[C_1 2t + C_6]\frac{\partial\Lambda}{\partial t} + C_2\frac{\partial\Lambda}{\partial\varphi} + C_1 r\frac{\partial\Lambda}{\partial r} + [C_1 z + C_3]\frac{\partial\Lambda}{\partial z} = 0\,.$$

By solving this equation by the method of characteristics [12], we find

$$\Lambda = \eta\,(t, r) = \frac{r/L}{\sqrt{\text{Fo} + C_6}}\,,\quad \Lambda = \xi\,(t, \varphi) = (\text{Fo} + C_6)\exp(b\varphi)\,,\quad \Lambda = \zeta\,(t, z) = \frac{z/L + C_3^*}{\sqrt{\text{Fo} + C_6}}\,,$$

where the dimensionless numbers Fo and $b = \text{const}$ are formed from the constant involved in (2) and time using the kinematic viscosity and the linear scale. Analogously we define self-similar functions. However, in defining, we must select a parametric variable which will be involved in the self-similar function as a coefficient. For example, if we select the time as the parametric variable, the governing equation for the radial velocity component will be as follows:

$$[C_1 2t + C_6]\frac{\partial U}{\partial t} - C_1 u\frac{\partial U}{\partial u} = 0\,.$$

The equations for all the remaining self-similar functions are constructed according to the same scheme. By solution of them we have

$$u\,(t, r, \varphi, z) = \frac{U(\eta, \xi, \zeta)}{\sqrt{\text{Fo} + C_6}}\,,\quad v\,(t, r, \varphi, z) = \frac{V(\eta, \xi, \zeta)}{\sqrt{\text{Fo} + C_6}}\,,\quad w\,(t, r, \varphi, z) = \frac{W(\eta, \xi, \zeta)}{\sqrt{\text{Fo} + C_6}}\,\nu\,,$$

$$p\,(t, r, \varphi, z) = \rho\,\frac{P\,(\eta, \xi, \zeta)}{\text{Fo} + C_6}\left(\frac{\nu}{L}\right)^2 + C_5\varepsilon z + C_4\varepsilon\tau\,(t)\,,\quad T\,(t, r, \varphi, z) = \Delta T\,\frac{\Theta\,(\eta, \xi, \zeta)}{\sqrt{(\text{Fo} + C_6)^3}} + \frac{C_5\varepsilon}{g\beta\rho}\,.$$

After the substitution of the self-similar variables into (1) we obtain

$$U\frac{\partial U}{\partial \eta} + b\,\frac{\xi}{\eta}\,V\frac{\partial U}{\partial \xi} + W\frac{\partial U}{\partial \zeta} - \frac{V^2}{\eta} = -\frac{\partial P}{\partial \eta} + \frac{\partial^2 U}{\partial \eta^2} + \frac{\partial U}{\partial \eta}\left(\frac{1}{\eta} + \frac{\eta}{2}\right) - U\left(\frac{1}{\eta^2} - \frac{1}{2}\right) +$$

$$+ b^2\,\frac{\xi^2}{\eta^2}\,\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial U}{\partial \xi}\left(b^2\,\frac{\xi}{\eta^2} - \xi\right) + \frac{\partial^2 U}{\partial \zeta^2} + \frac{\zeta}{2}\,\frac{\partial U}{\partial \zeta} - 2b\,\frac{\xi}{\eta^2}\,\frac{\partial V}{\partial \xi}\,,$$

$$U\frac{\partial V}{\partial \eta} + b\,\frac{\xi}{\eta}\,V\frac{\partial V}{\partial \xi} + W\frac{\partial V}{\partial \zeta} + \frac{VU}{\eta} = -\,b\,\frac{\xi}{\eta}\,\frac{\partial P}{\partial \xi} + \frac{\partial^2 V}{\partial \eta^2} + \frac{\partial V}{\partial \eta}\left(\frac{1}{\eta} + \frac{\eta}{2}\right) - V\left(\frac{1}{\eta^2} - \frac{1}{2}\right) +$$

$$+ b^2\,\frac{\xi^2}{\eta^2}\,\frac{\partial^2 V}{\partial \xi^2} + \frac{\partial V}{\partial \xi}\left(b^2\,\frac{\xi}{\eta^2} - \xi\right) + \frac{\partial^2 V}{\partial \zeta^2} + \frac{\zeta}{2}\,\frac{\partial V}{\partial \zeta} + 2b\,\frac{\xi}{\eta^2}\,\frac{\partial U}{\partial \xi}\,,$$

$$U\frac{\partial W}{\partial \eta} + b\,\frac{\xi}{\eta}\,V\frac{\partial W}{\partial \xi} + W\frac{\partial W}{\partial \zeta} = -\frac{\partial P}{\partial \zeta} + \frac{\partial^2 W}{\partial \eta^2} + \frac{\partial W}{\partial \eta}\left(\frac{1}{\eta} + \frac{\eta}{2}\right) + \frac{W}{2} +$$

$$+ b^2\,\frac{\xi^2}{\eta^2}\,\frac{\partial^2 W}{\partial \xi^2} + \frac{\partial W}{\partial \xi}\left(b^2\,\frac{\xi}{\eta^2} - \xi\right) + \frac{\partial^2 W}{\partial \zeta^2} + \frac{\zeta}{2}\,\frac{\partial W}{\partial \zeta} + \text{Gr}\,\Theta\,,$$

$$\frac{\partial U}{\partial \eta} + \frac{U}{\eta} + b\,\frac{\xi}{\eta}\,\frac{\partial V}{\partial \xi} + \frac{\partial W}{\partial \zeta} = 0\,,$$

$$\text{Pr}\left(U\frac{\partial \Theta}{\partial \eta} + b\,\frac{\xi}{\eta}\,V\frac{\partial \Theta}{\partial \xi} + W\frac{\partial \Theta}{\partial \zeta}\right) = \frac{\partial^2 \Theta}{\partial \eta^2} + \frac{\partial \Theta}{\partial \eta}\left(\frac{1}{\eta} + \frac{\eta\,\text{Pr}}{2}\right) + \frac{3\,\text{Pr}\,\Theta}{2} +$$

$$+ b^2\,\frac{\xi^2}{\eta^2}\,\frac{\partial^2 \Theta}{\partial \xi^2} + \frac{\partial \Theta}{\partial \xi}\left(b^2\,\frac{\xi}{\eta^2} - \text{Pr}\,\xi\right) + \frac{\partial^2 \Theta}{\partial \zeta^2} + \frac{\text{Pr}\,\zeta}{2}\,\frac{\partial \Theta}{\partial \zeta}\,.$$

In the case of a two-dimensional flow in the plane r–z we can decrease the number of dependent variables. To do this we represent system (1) in the form of the equation for vorticity

$$\frac{\partial \Omega}{\partial t} + u\,\frac{\partial \Omega}{\partial r} + w\,\frac{\partial \Omega}{\partial z} - u\,\frac{\Omega}{r} = \nu\left(\frac{\partial^2 \Omega}{\partial r^2} + \frac{1}{r}\,\frac{\partial \Omega}{\partial r} - \frac{\Omega}{r^2} + \frac{\partial^2 \Omega}{\partial z^2}\right) - g\beta\,\frac{\partial T}{\partial r}\,, \tag{3}$$

where

$$\Omega = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\,.$$

If we substitute the self-similar variables into (3), we obtain an equation not containing pressure. Thus, we decrease the number of dependent variables by one. If we introduce the current function Ψ satisfying the conditions

$$U = \frac{1}{\eta}\,\frac{\partial \Psi}{\partial \zeta}\,,\quad W = -\,\frac{1}{\eta}\,\frac{\partial \Psi}{\partial \eta}\,,$$

we can decrease the number of dependent variables by two, considering η and ζ to be independent quantities.

The form of the self-similar variables proposed is not always convenient for calculations, since the parametric variable and all the self-similar variables contain time. For steady-state flows it is more convenient to employ one coordinate as the parametric variable. For a cylindrical system, unlike the Cartesian one, selection of a new parametric variable automatically means a new form of self-similar equations. Selecting a new parametric variable and new forms of self-similar variables, we can exhaust all the variants and obtain rather cumbersome final expressions. All possible self-similar forms are easy to implement by employing the MATHEMATICA package.

Let us pass to an analysis of two-dimensional flow where the problem can be reduced to a system of ordinary differential equations. For the flow in the plane r–z we introduce the following variables:

$$\zeta(r, z) = \frac{r}{z + C_3}, \quad w(t, r, \varphi, z) = \frac{df(\zeta)}{d\zeta} \frac{\nu}{(z + C_3)\,\zeta}, \quad T(t, r, \varphi, z) = \Delta T \frac{\Theta(\zeta)}{(z + C_3)^3} L^3,$$

having selected the quantity $(z + C_3)$ as the parametric variable. We obtain the radial component of the velocity on the basis of the continuity equation and then substitute the expressions obtained into Eq. (1). As a result we have

$$f^{\mathrm{IV}}[\zeta^3 + 2\zeta^5 + \zeta^7] + f'''[f(\zeta^2 + \zeta^4) - 2\zeta^2 + 6\zeta^4 + 8\zeta^6] + 3f''[f'(\zeta^2 + \zeta^4) - f\zeta + \zeta + 4\zeta^5] -$$

$$- 3f'[1 - f + f'\zeta] + \mathrm{Gr}\,\Theta\zeta^4 = 0,$$

$$\Theta''[\zeta + \zeta^3] + \Theta'[1 + 8\zeta^2\,\mathrm{Pr}\,f] + 3\Theta[4\zeta + \mathrm{Pr}\,f'] = 0.$$

For this self-similar form it is easy to implement a numerical algorithm of calculation (boundary-value problem) on the basis of the MATHCAD package. This form contains no pressure. By employing Eq. (1) one can easily obtain self-similar forms including pressure.

If we select the radial component as the parametric variable, self-similar forms appear as

$$\zeta(r, z) = \frac{z + C_3}{r}, \quad u(t, r, \varphi, z) = \frac{df(\zeta)}{d\zeta} \frac{\nu}{r}, \quad T(t, r, \varphi, z) = \Delta T \frac{\Theta(\zeta)}{r^3} L^3,$$

$$f^{\mathrm{IV}}[1 + 2\zeta^2 + \zeta^4] + f'''[f(1 + \zeta^2) + 10\zeta + 10\zeta^3) + 3f''[f'(1 + \zeta^2) + f\zeta + 7\zeta^2 + 6] +$$

$$+ 3f'[\zeta - f + f'\zeta] - 3f + \mathrm{Gr}\,[3\Theta + \Theta'\zeta] = 0,$$

$$\Theta''[\zeta^4 + \zeta^6] + \Theta'[7\zeta^5 + \mathrm{Pr}\,\zeta^4 f] + 3\Theta[3 + \mathrm{Pr}\,f']\,\zeta^4 = 0.$$

The symmetry properties enable us to obtain the forms of self-similar variables characteristic of wave processes. If we employ the symmetries $q_{2,3,6}$ from (2), we obtain the following self-similar variables:

$$\eta(t, \varphi, z) = k_\varphi \varphi + k_z z - \omega t, \quad \zeta = r/L,$$

$$u(t, r, \varphi, z) = U(\eta, \zeta)\,\nu/L, \quad v(t, r, \varphi, z) = V(\eta, \zeta)\,\nu/L, \quad w(t, r, \varphi, z) = W(\eta, \zeta)\,\nu/L,$$

$$p(t, r, \varphi, z) = P(\eta, \zeta)\,\nu^2\rho/L^2, \quad T(t, r, \varphi, z) = \Delta T\,\Theta(\eta, \zeta).$$

Using them we can transform expressions (1) to the form

$$\frac{\partial U}{\partial \zeta} + \frac{U}{\zeta} + \frac{k_\varphi}{\zeta}\frac{\partial V}{\partial \eta} + \bar{k}_z \frac{\partial W}{\partial \eta} = 0,$$

$$U\frac{\partial U}{\partial \zeta}+\left(\frac{k_\varphi V}{\zeta}+\bar{k}_z W-\mathrm{Fo}_\Omega\right)\frac{\partial U}{\partial \eta}-\frac{V^2}{\zeta}=-\frac{\partial P}{\partial \zeta}+\frac{\partial^2 U}{\partial \zeta^2}+\frac{1}{\zeta}\frac{\partial U}{\partial \zeta}-\frac{U}{\zeta^2}+\left(\bar{k}_z^2+\frac{k_\varphi^2}{\zeta^2}\right)\frac{\partial^2 U}{\partial \eta^2}-\frac{2k_\varphi}{\zeta^2}\frac{\partial V}{\partial \eta},$$

$$U\frac{\partial V}{\partial \zeta}+\left(\frac{k_\varphi V}{\zeta}+\bar{k}_z W-\mathrm{Fo}_\Omega\right)\frac{\partial V}{\partial \eta}+\frac{UV}{\zeta}=-\frac{k_\varphi}{\zeta}\frac{\partial P}{\partial \eta}+\frac{\partial^2 V}{\partial \zeta^2}+\frac{1}{\zeta}\frac{\partial V}{\partial \zeta}-\frac{V}{\zeta^2}+\left(\bar{k}_z^2+\frac{k_\varphi^2}{\zeta^2}\right)\frac{\partial^2 V}{\partial \eta^2}+\frac{2k_\varphi}{\zeta^2}\frac{\partial U}{\partial \eta},$$

$$U\frac{\partial W}{\partial \zeta}+\left(\frac{k_\varphi V}{\zeta}+\bar{k}_z W-\mathrm{Fo}_\Omega\right)\frac{\partial W}{\partial \eta}=-\bar{k}_z\frac{\partial P}{\partial \eta}+\frac{\partial^2 W}{\partial \zeta^2}+\frac{1}{\zeta}\frac{\partial W}{\partial \zeta}+\left(\bar{k}_z^2+\frac{k_\varphi^2}{\zeta^2}\right)\frac{\partial^2 W}{\partial \eta^2}+\mathrm{Gr}\,\Theta,$$

$$\left(\mathrm{Pr}\,U-\frac{1}{\zeta}\right)\frac{\partial \Theta}{\partial \zeta}+\mathrm{Pr}\left(\frac{k_\varphi V}{\zeta}+\bar{k}_z W-\mathrm{Fo}_\Omega\right)\frac{\partial \Theta}{\partial \eta}=\frac{\partial^2 \Theta}{\partial \zeta^2}+\frac{1}{\zeta}\frac{\partial W}{\partial \zeta}+\left(\bar{k}_z^2+\frac{k_\varphi^2}{\zeta^2}\right)\frac{\partial^2 \Theta}{\partial \eta^2},$$

where $\bar{k}_z = k_z L$. If we consider the hypothetical case where the wave is propagating in the cylindrical plane $r = \mathrm{const}$ in the absence of the radial component, we can obtain an analytical solution. With the indicated assumptions (without violating the generality of considerations we take $\zeta = 1$) the previous system is transformed to the form

$$k_\varphi\frac{dV}{d\eta}+\bar{k}_z\frac{dW}{d\eta}=0,$$

$$(k_\varphi V+\bar{k}_z W-\mathrm{Fo}_\Omega)\frac{dV}{d\eta}=-k_\varphi\frac{dP}{d\eta}+k^2\frac{d^2 V}{d\eta^2},$$

$$(k_\varphi V+\bar{k}_z W-\mathrm{Fo}_\Omega)\frac{dW}{d\eta}=-\bar{k}_z\frac{dP}{d\eta}+k^2\frac{d^2 W}{d\eta^2}+\mathrm{Gr}\,\Theta,$$

$$\mathrm{Pr}\,(k_\varphi V+\bar{k}_z W-\mathrm{Fo}_\Omega)\frac{d\Theta}{d\eta}=k^2\frac{d^2 \Theta}{d\eta^2},$$

where $\bar{k}_z^2 + k_\varphi^2 = k^2$.

From the continuity equation we find

$$k_\varphi V+\bar{k}_z W=C^*=\mathrm{const}.$$

By employing this relation, we transform the previous system of equations as

$$\mathrm{Pr}\,(C^*-\mathrm{Fo}_\Omega)\frac{d\Theta}{d\eta}=k^2\frac{d^2 \Theta}{d\eta^2},$$

$$k^2\frac{dP}{d\eta}=\bar{k}_z\,\mathrm{Gr}\,\Theta,$$

$$(C^*-\mathrm{Fo}_\Omega)\frac{dV}{d\eta}=-k_\varphi\frac{dP}{d\eta}+k^2\frac{d^2 V}{d\eta^2},$$

$$(C^*-\mathrm{Fo}_\Omega)\frac{dW}{d\eta}=-\bar{k}_z\frac{dP}{d\eta}+k^2\frac{d^2 W}{d\eta^2}+\mathrm{Gr}\,\Theta.$$

This system is easily solved in the form [12]

$$\Theta = c_2 + \frac{c_5 k^2}{(C^* - \mathrm{Fo}_\Omega)\,\mathrm{Pr}}\left[\exp\left(\frac{(C^* - \mathrm{Fo}_\Omega)\,\mathrm{Pr}\,\eta}{k^2}\right) - 1\right],$$

$$P = c_1 + \bar{k}_z\,\mathrm{Gr}\left[\frac{c_5 k^2}{(C^* - \mathrm{Fo}_\Omega)^2\,\mathrm{Pr}^2}\left[\exp\left(\frac{(C^* - \mathrm{Fo}_\Omega)\,\mathrm{Pr}\,\eta}{k^2}\right) - 1\right] + \eta\left(\frac{c_2}{k^2} - \frac{c_5}{(C^* - \mathrm{Fo}_\Omega)\,\mathrm{Pr}}\right)\right],$$

$$V = c_3 + \frac{c_6 k^2}{C^* - \mathrm{Fo}_c}\left[\exp\left(\frac{(C^* - \mathrm{Fo}_c)\,\eta}{k^2}\right) - 1\right] - \frac{k_\varphi \bar{k}_z\,\mathrm{Gr}}{k^2\,(C^* - \mathrm{Fo}_c)^3}\left\{c_2\,(C^{*2} + \mathrm{Fo}_c^2)\,\eta + \frac{c_5 k^4}{\mathrm{Pr}^2\,(\mathrm{Pr} - 1)}\times\right.$$

$$\times\left[\left(\exp\left(\frac{(C^* - \mathrm{Fo}_c)\,\mathrm{Pr}\,\eta}{k^2}\right) - 1\right) - \mathrm{Pr}^2\left(\exp\left(\frac{(C^* - \mathrm{Fo}_c)\,\eta}{k^2}\right) - 1\right)\right] +$$

$$+ \frac{k^2\,\mathrm{Fo}_c}{\mathrm{Pr}}\left[c_2\,\mathrm{Pr}\left(\exp\left(\frac{(C^* - \mathrm{Fo}_c)\,\eta}{k^2}\right) - 1\right) + c_5 \eta\right] +$$

$$\left. + \frac{C^*}{\mathrm{Pr}}\left[\eta\,(c_5 + 2c_2\,\mathrm{Fo}_c\,\mathrm{Pr}) + c_2 k^2\,\mathrm{Pr}\left(\exp\left(\frac{(C^* - \mathrm{Fo}_c)\,\eta}{k^2}\right) - 1\right)\right]\right\},$$

$$W = c_4 + \frac{c_7 k^2}{C^* - \mathrm{Fo}_\Omega}\left[\exp\left(\frac{(C^* - \mathrm{Fo}_\Omega)\,\eta}{k^2}\right) - 1\right] + \frac{k_\varphi^2\,\mathrm{Gr}}{k^2\,(C^* - \mathrm{Fo}_\Omega)^3}\left\{c_2\,(C^{*2} + \mathrm{Fo}_\Omega^2)\,\eta - \frac{c_5 k^4}{\mathrm{Pr}^2\,(\mathrm{Pr} - 1)}\times\right.$$

$$\times\left[\left(\exp\left(\frac{(C^* - \mathrm{Fo}_\Omega)\,\mathrm{Pr}\,\eta}{k^2}\right) - 1\right) - \mathrm{Pr}^2\left(\exp\left(\frac{(C^* - \mathrm{Fo}_\Omega)\,\eta}{k^2}\right) - 1\right)\right] +$$

$$+ \frac{k^2\,\mathrm{Fo}_\Omega}{\mathrm{Pr}}\left[c_2\,\mathrm{Pr}\left(\exp\left(\frac{(C^* - \mathrm{Fo}_\Omega)\,\eta}{k^2}\right) - 1\right) + c_5 \eta\right] -$$

$$\left. - \frac{C^*}{\mathrm{Pr}}\left[\eta\,(c_5 + 2c_2\,\mathrm{Fo}_\Omega\,\mathrm{Pr}) + c_2 k^2\,\mathrm{Pr}\left(\exp\left(\frac{(C^* - \mathrm{Fo}_\Omega)\,\eta}{k^2}\right) - 1\right)\right]\right\}.$$

The introduction of complexity into the constant coefficients yields a wave character of the solution.

In closing, we consider discrete symmetries. The discrete symmetries differ from the classical Lie point symmetries in the fact that they are true just for the discrete values of the transformation parameter ε. In our case we have the transformations of a mirror image where the coordinates and corresponding velocity components and the temperature change their sign to the opposite one simultaneously. The symmetry C_7 yields the following transformations:

$$w \to w \exp(\varepsilon)\,, \quad z \to z \exp(\varepsilon)\,, \quad T \to T \exp(\varepsilon)\,,$$

obtained based in (2) according to [1]. But if these transformations are applied to system (1), we will see that they are true just for $\varepsilon = i(\pi + n\pi)$ where $n = 0, 2, 4, \ldots$, i.e., for discrete values of ε. However this does not interfere in determining self-similar forms using the infinitesimal generator (2). A distinctive feature of discrete symmetries is that they do not agree with classical symmetries in constructing self-similar variables. Therefore, now we will employ only those terms of (2) which have the coefficient C_7. In this case, only z can be selected as the parametric variable. As a result we arrive at the following relations for the self-similar forms:

$$\eta\,(t) = \mathrm{Fo}\,, \quad \xi\,(\varphi) = \varphi\,, \quad \zeta\,(r) = \frac{r}{L}\,,$$

$$u\,(t, x, y, z) = \frac{U\,(\eta, \xi, \zeta)}{L}\,\nu\,, \quad v\,(t, x, y, z) = \frac{V\,(\eta, \xi, \zeta)}{L}\,\nu\,, \quad w\,(t, x, y, z) = \frac{W\,(\eta, \xi, \zeta)}{L^2}\,\nu z\,,$$

$$p\,(t, x, y, z) = \frac{P\,(\eta, \xi, \zeta)}{L^2}\,\rho\nu^2\,, \quad T\,(t, x, y, z) = \Delta T\Theta\,(\eta, \xi, \zeta)\,\frac{z}{L}\,.$$

They make it possible to obtain, from (1), the following self-similar equations:

$$\frac{\partial U}{\partial \zeta} + \frac{U}{\zeta} + W + \frac{\partial V}{\partial \xi} = 0\,,$$

$$\frac{\partial U}{\partial \eta} + U\frac{\partial U}{\partial \zeta} + \frac{V}{\zeta}\frac{\partial U}{\partial \xi} - \frac{V^2}{\zeta} = -\frac{\partial P}{\partial \zeta} + \frac{\partial^2 U}{\partial \zeta^2} + \frac{1}{\zeta}\frac{\partial U}{\partial \zeta} - \frac{U}{\zeta^2} + \frac{1}{\zeta^2}\frac{\partial^2 U}{\partial \xi^2} - \frac{2}{\zeta^2}\frac{\partial V}{\partial \xi}\,,$$

$$\frac{\partial V}{\partial \eta} + U\frac{\partial V}{\partial \zeta} + \frac{V}{\zeta}\frac{\partial V}{\partial \xi} + \frac{VU}{\zeta} = -\frac{1}{\zeta}\frac{\partial P}{\partial \xi} + \frac{\partial^2 V}{\partial \zeta^2} + \frac{1}{\zeta}\frac{\partial V}{\partial \zeta} - \frac{V}{\zeta^2} + \frac{1}{\zeta^2}\frac{\partial^2 V}{\partial \xi^2} + \frac{2}{\zeta^2}\frac{\partial U}{\partial \xi}\,,$$

$$\frac{\partial W}{\partial \eta} + U\frac{\partial W}{\partial \zeta} + \frac{V}{\zeta}\frac{\partial W}{\partial \xi} + W^2 = \frac{\partial^2 W}{\partial \zeta^2} + \frac{1}{\zeta}\frac{\partial W}{\partial \zeta} + \frac{1}{\zeta^2}\frac{\partial^2 W}{\partial \xi^2} + \mathrm{Gr}\,\Theta\,,$$

$$\mathrm{Pr}\left(\frac{\partial \Theta}{\partial \eta} + U\frac{\partial \Theta}{\partial \zeta} + \frac{V}{\zeta}\frac{\partial \Theta}{\partial \xi} + W\Theta\right) = \frac{\partial^2 \Theta}{\partial \zeta^2} + \frac{1}{\zeta}\frac{\partial \Theta}{\partial \zeta} + \frac{1}{\zeta^2}\frac{\partial^2 \Theta}{\partial \xi^2}\,.$$

In the case of the two-dimensional flow in the plane r–z we can reduce the problem to a system of ordinary differential equations if we introduce the variables

$$w = \frac{\nu z}{\zeta L^2}\,f'\,(\zeta)\,, \quad u = -\frac{\nu}{\zeta L}\,f\,(\zeta)\,, \quad p = \frac{\nu^2\rho}{L^2}\,P\,(\zeta)\,, \quad T = \frac{z}{L}\,\Theta\,(\zeta)\,.$$

Application of the latter to system (1) yields

$$-\zeta\,\mathrm{Gr}\,\Theta + \frac{ff' - f'}{\zeta^2} + \frac{f'^2 - ff'' + f''}{\zeta} = f'''\,, \quad \frac{f''}{\zeta} + \frac{ff' - f'}{\zeta^2} - \frac{f^2}{\zeta^3} + P' = 0\,, \quad \frac{\mathrm{Pr}\,(\Theta f' - f\Theta') - \Theta'}{\zeta} = \Theta''\,.$$

If the self-similar variable is introduced in the manner below

$$\zeta = \frac{r\,(g\beta\Delta T)^{1/3}}{\nu^{2/3}}\,, \tag{4}$$

the Grashof number in the previous system is abolished. If the pressure is eliminated, we will have

$$\frac{f^{\mathrm{IV}}}{\zeta} + \mathrm{Gr}\,\Theta' + \frac{ff''' - f'f'' - 2f'''}{\zeta^2} + \frac{f'^2 - 3ff'' + 3f''}{\zeta^3} + \frac{3ff' - 3f'}{\zeta^4} = 0\,, \quad \frac{\mathrm{Pr}\,(\Theta f' - f\Theta') - \Theta'}{\zeta} = \Theta''$$

or the analogous equation without a Grashof number when (4) is employed.

In the case where the flow is independent of the tangential coordinate, system (1) possesses another discrete symmetry:

$$q = r\partial_r + u\partial_u + v\partial_v .$$

The latter has been employed to analyze the flow [13] and heat exchange [14] near a rotating disk. In the absence of the tangential component of the velocity, we can introduce the self-similar forms

$$w = -\frac{2\nu}{L} f(\eta) , \quad u = \frac{\nu r}{L^2} f'(\eta) , \quad p = \frac{\nu^2 \rho}{L^2} P(\eta) , \quad T = \Theta(\eta) , \quad \eta = \frac{z}{L} ,$$

that generate the system of equations

$$-\,\mathrm{Gr}\,\Theta + 4ff' + P' + 2f'' = 0 , \quad f''' = f'^2 - 2ff'' , \quad -\,2\,\mathrm{Pr}\,f\Theta' = \Theta'' .$$

The equation of heat exchange can be solved in quadratures. We can also eliminate the pressure from the last system and obtain an equation of fourth order.

In the present investigation, we have analyzed the symmetries and corresponding self-similar forms for the system of Navier–Stokes equations and equations of convective heat exchange in a cylindrical coordinate system. It has been shown how one can obtain self-similar forms for variables and equations by employing the apparatus of the theory of Lie groups [1] on the basis of the symmetries of differential equations. From the above analysis it follows that extension symmetries are the most attractive from the viewpoint of construction of self-similar forms. This is also confirmed by investigations for parabolic flows of the type of a boundary layer [2–7]. Extension symmetries make it possible to generate self-similar forms convenient to solve boundary-value problems either analytically or numerically. Numerical investigation on the basis of modern application packages (for example, MATHCAD) presents no difficulties [4]. Elliptical equations, such as complete Navier–Stokes equations, most often possess symmetries which are also true for truncated parabolic analogs, such as Prandtl equations for the boundary layer. Therefore, the symmetries of complete elliptical equations can be employed to construct the self-similar forms of parabolic flows. Even though the equation does not correspond in full measure to one symmetry or another (because of the presence of additional terms which characterize the influence of disturbing factors: the longitudinal pressure gradient, injection, suction, curvature of streamlines, incompressibility, etc.), one can employ this symmetry to construct partially self-similar equations [3]. And these equations can be integrated at each step of a marching variable as a system of ordinary differential equations.

Transfer symmetries are convenient for description of wave processes. They are characteristic of systems of equations not containing unknown arguments in explicit form. Wave self-similar forms are also convenient in investigating the processes of hydrodynamic instability since they enable one to represent wave-disturbing solutions not only as one-dimensional waves (Tollmien–Schlichting waves) but also as two- and three-dimensional waves.

In the work, we have also studied nonclassical discrete symmetries that cannot be determined by a standard method of investigation [1]. The discrete symmetries differ from the classical ones in the fact that they are true only for the discrete values of the transformation parameter. These symmetries have been employed by some authors (see [13]) to construct self-similar forms at an intuitive level without rigorous theoretical substantiations.

The main difficulty of the proposed method of investigation is to relate a specific self-similar form (or symmetry which generates this form) to the boundary-value problem under study. Different symmetries generate different self-similar forms of one and the same system of partial differential equations. There is no unambiguous rigorous criterion relating a given self-similar form to boundary conditions which must be satisfied in a specific problem. Therefore, in each specific case it is necessary to find such a criterion from additional conditions. In parabolic jet flows, in flows in a wake behind a body [13], and in other problems [15, 16], one employs additional integral conditions of constancy of the resistance force, the momentum, or the moment of momentum. From the constancy of the quantities mentioned one selects the form of self-similar variables which can be obtained on the basis of the symmetries of the process under study. However, most frequently researchers rely on intuition in finding the form of self-similar variables as, for example, H. Blasius [13] in solving the classical problem on the boundary layer near a plane surface.

Self-similar forms also find application in experimental investigations, since they immediately indicate in what coordinates one should process experimental data. Such a processing of experimental points can provide an answer whether the process under study is self-similar. This can also demonstrate the correctness of a mathematical model of the process.

NOTATION

a, thermal diffusivity; *g*, free-fall acceleration; k_2, k_φ, and *k*, wave numbers; *L*, characteristic dimension (linear scale); *p*, pressure; *q*, infinitesimal generator; *r*, φ, *z*, cylindrical coordinates; *t*, time; *T*, temperature; Δ*T*, characteristic difference (temperature scale); *u*, *v*, *w*, velocity components corresponding to the coordinates *r*, φ, *z*; β, coefficient of thermal expansion; τ(*t*), arbitrary time function; ε, parameter of group transformation; η, ξ, and ζ, self-similar variables; ν, kinematic viscosity; ρ, density; ω, frequency; Ω, vorticity; Ψ, current function; *f*, self-similar current function; $\mathrm{Fo} = t\nu/L^2$ and $\mathrm{Fo}_\Omega = \omega L^2/\nu$, Fourier numbers; $\mathrm{Gr} = g\beta\Delta T L^3/\nu^2$, Grashof numbers; *P*, self-similar pressure function; $\mathrm{Pr} = \nu/a$, Prandtl number; *U*, *V*, and *W*, self-similar functions of the velocity component corresponding to *u*, *v*, *w*; Θ, self-similar temperature function.

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